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An inventory model for single-period products with reordering opportunities under fuzzy demand

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Abstract

This paper analyzes a single-period inventory model of profit maximization with a reordering strategy in an imprecise environment. The entire period is divided into two slots and the customer demand is considered as a fuzzy number in situations where the demand in each slot is linguistic in nature and characterized as '*demand is about d*'. The reordering is to be done during the mid-season after the early-season demand has been observed. The objective is to determine the optimal order quantity in maximizing the expected resultant profit by considering the fuzzy demand and reordering strategy in the single-period framework. The solution procedure is presented using ordering of fuzzy numbers with respect to their possibilistic mean values. Numerical examples are given to illustrate the efficiency of this strategy.

Keywords: Single-period inventory; Fuzzy demand; Reordering strategy; Profit maximization

1. Introduction

In the present day competitive business scenario, while dealing with single-period products the basic problem is how best one can manage its inventory. In traditional single-period inventory models a business strategy is described where only a single procurement is made for a specific period under probabilistic demand. It is well known that in developing inventory models, the major difficulty faced by the decision maker is that of forecasting the demand. In the literature, several models on single-period products assume stochastic demand [1,2]. But, in the present day scenario, it is difficult to decide the exact demand, i.e., how many items customers will purchase during the whole season/period. Considering this view, several researchers developed fuzzy inventory models for situations where the customer demand is described linguistically, like "demand is about d" [3–6]. However, either due to lack of evidence or uncertainty in judgment, to model a realistic decision-making inventory problem the customer demand can be described by a knowledge-based uncertainty. This paper adopts the fuzziness of customer demand explicitly into the single-period model together with the reordering strategy.

Petrovic et al. [4] first proposed a newsboy-type problem with discrete fuzzy demand. Li et al. [5] and Kao and Hsu [6] also investigated the single-period inventory model under a fuzzy environment where the optimization

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is achieved through ordering fuzzy numbers using their total integral values. In one of our earlier works [7] we incorporated fuzzy random variable as customer demand and developed the model in a mixed imprecise and uncertain environment. Chen and Chuang [8] considered an extended newsboy problem with shortage-level constraints. Khouja and Robbins [9] considered the effects of advertisement in the single-period framework, which leads to increase the sales. Ben-Daya and Raouf [10] presented constrained multi-item single-period inventory models. Recently, Layek et al. [11] proposed an exact, approximate, and generic iterative model for the multi-product newsboy problem with budget constraint. They introduced a generic iterative method to develop an exact, approximate solution procedure where the demand distribution for each product may occur in different manners. In [12] Layek and Montanari investigated a dual solution space for a multi-item newsboy problem with constraints.

However, in all these studies it is assumed that only one order can be placed for a specific period. But in practice, this does not always meet the decision maker's (DM's)/retailer's achievement as well as the customer satisfaction (e.g., a shopkeeper cannot purchase an item largely due to the constraints though there is a demand). Thus if there is an opportunity for reordering during the middle of the season then that would not only reduce the initial investment of money but also reduce the storage space problem. Practically, in the rapid communication age of business, there exist several single-period products (e.g., fashion/seasonal goods, perishable items etc.) that need replenishment later in the season/period either due to limited capacity or demand uncertainty.

In the present investigation we attempt to focus on a two-ordering strategy throughout the whole season, where the reordering is to be done during the mid-season after the early-season demand has been observed. Lau and Lau [13,14] first proposed a reordering strategy for a newsboy-type problem with stochastic demand. They divided the whole season into two slots by means of a demand scale and allowed shortages for both the slots. We develop the reordering strategy with fuzzy demand where the slots are characterized by the time scale rather than the demand scale. In real situations, a spot seller/spot retailer who sells bread, fish or some seasonal items may have the option for reordering during the middle of the period, but after the end of the season/period items are either abolished or obsolete.

In the present work, we analyze, if there is an opportunity for reordering, what will be the optimal policy to pursue in order that the expected optimal order quantity can be found by maximizing its profit function in a fuzzy environment. As the demand is linguistic in nature and the optimal order quantity in the second slot depends upon the leftover items from slot 1, the decision variable during the second slot is clearly a fuzzy quantity. To determine the optimal order quantity that maximizes the profit function we use the possibilistic mean value of a fuzzy number to rank fuzzy numbers [15].

The paper is organized as follows. In Section 2 we give some basic definitions and notation centered on our present work. The proposed fuzzy model is described in Section 3 and this is followed by some numerical examples in Section 4. Finally in Section 5 we give a few concluding remarks.

2. Preliminaries

In this section, we shall define some basic definitions and notation relevant for understanding the fuzzy inventory model proposed in this paper.

2.1. Fuzzy number and its features

Definition 1. A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\Re$ with a normal, convex and continuous membership function of bounded support.

A triangular distributed fuzzy number $\tilde{A} = (\underline{A}, A, \overline{A})$ can be described with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(x) = \left(\dfrac{x - \underline{A}}{A - \underline{A}}\right), & \underline{A} \le x \le A \\[2ex] R(x) = \left(\dfrac{\overline{A} - x}{\overline{A} - A}\right), & A \le x \le \overline{A} \end{cases}$$

where A is the modal of fuzzy number $\tilde{A}$, $L, R : \Re \to [0, 1]$ are the left and right shape continuous functions. Hence the closure of the support of $\tilde{A}$ is exactly $[\underline{A}, \overline{A}]$.

For a given fuzzy set $\tilde{A}$, $A(\alpha)$, the alpha-cut set, is given by $A(\alpha) = \{x/\mu_{\tilde{A}}(x) \ge \alpha\}$ and is denoted by the interval $[A_L(\alpha), A_R(\alpha)]$, where $0 \le \alpha \le 1$.

Definition 2. The interval-valued expectation of $\tilde{A}$ is defined as $E(\tilde{A}) = [E_*(\tilde{A}), E^*(\tilde{A})]$ where $E_*(\tilde{A}) = \int_0^1 A_L(\alpha)\mathrm{d}\alpha$ and $E^*(\tilde{A}) = \int_0^1 A_R(\alpha)\mathrm{d}\alpha$ are the left and right integral values of $\tilde{A}$, respectively [16,17].

The expected mean value of $\tilde{A}$ based on the area measurement index is defined as

$$\overline{E}(\tilde{A}) = \frac{E_*(\tilde{A}) + E^*(\tilde{A})}{2}.$$

Definition 3. The interval-valued possibilistic mean is defined as $M(\tilde{A}) = [M_*(\tilde{A}), M^*(\tilde{A})]$ where $M_*(\tilde{A})$ and $M^*(\tilde{A})$ are the lower and upper possibilistic mean values of $\tilde{A}$ [15] and are respectively defined by

$$M_*(\tilde{A}) = \frac{\int_0^1 \alpha A_L(\alpha)\mathrm{d}\alpha}{\int_0^1 \alpha \mathrm{d}\alpha}, \qquad M^*(\tilde{A}) = \frac{\int_0^1 \alpha A_R(\alpha)\mathrm{d}\alpha}{\int_0^1 \alpha \mathrm{d}\alpha}.$$

For a given fuzzy number $\tilde{A}$, it can be shown that $M(\tilde{A}) \subseteq E(\tilde{A})$.

The possibilistic mean value of $\tilde{A}$ is defined as

$$\overline{M}(\tilde{A}) = \frac{M_*(\tilde{A}) + M^*(\tilde{A})}{2}.$$

In other words, one can write $\overline{M}(\tilde{A}) = \int_0^1 \alpha(A_L(\alpha) + A_R(\alpha))\mathrm{d}\alpha$.

In applications, let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$ and $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$, $\alpha \in [0, 1]$; then for ranking fuzzy numbers, $\tilde{A} \le \tilde{B} \Leftrightarrow \overline{M}(\tilde{A}) \le \overline{M}(\tilde{B})$.

Definition 4. The graded mean integration representation of a triangular-based fuzzy number Ã with grade w is

$$G(\tilde{A}) = \int_0^w \alpha \left(\frac{L^{-1}(\alpha) + R^{-1}(\alpha)}{2}\right) \mathrm{d}\alpha \Big/ \int_0^w \alpha \mathrm{d}\alpha,$$

where α lies between 0 and w, $0 < w \le 1$. Here L^{-1} and R^{-1} are the inverse functions of L and R, respectively [18]. We shall use this definition in Appendix B.

2.2. Notation

To find the optimal policy for a single-period inventory model with two ordering strategies under imprecise demand information the following notation is used in this paper.

- c unit purchase cost
- p selling price per unit
- Q_1 order quantity at the beginning of the season/period
- Q_2 order quantity at the beginning of the second slot
- $\tilde{D}_1$ fuzzy demand for slot 1 (expert's opinion)
- $\tilde{D}_2$ fuzzy demand for slot 2 (expert's opinion)
- d_1 actual demand occurs in slot 1
- d_2 actual demand occurs in slot 2
- τ unsatisfied charge/penalty charge for unit unsold item in slot 1
- h holding cost per unit for the next season/period
- s_1 shortage cost per unit during slot 1
- s_2 shortage cost per unit during slot 2.

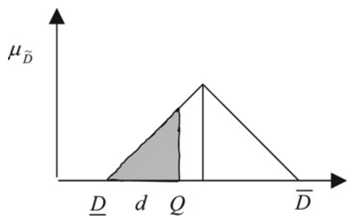


Fig. 1. For d lying in the range between $\underline{D}$ and Q, the shaded portion is to be taken into account as the membership function of $\tilde{D}$.

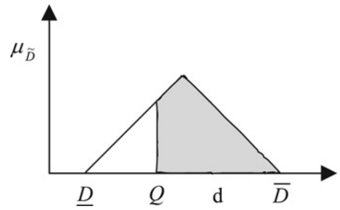


Fig. 2. For d lying in the range between Q and $\overline{D}$, the shaded portion is to be taken into account as the membership function of $\tilde{D}$.

2.3. Summary of the basic fuzzy model

The conventional, single-period products model with stochastic demand can be characterized as the well-known 'newsboy problem' where a single procurement is made at the beginning of the period. Usually the demand distribution is derived from evidence recorded in the past. But it is not always possible to forecast the exact information either due to the lack of evidence or uncertainty in judgment. Thus if the customer demand is characterized by a fuzzy number $\tilde{D}$ let Q be the given order quantity. As the demand $\tilde{D}$ is imprecisely prescribed, it may cause either a fuzzy overstock profit ($\widetilde{OP}$) if there is any excess or a fuzzy understock profit ($\widetilde{UP}$) for unsatisfied demand.

Mathematically, if d is the actual realization of demand then it can be written as

$$P(Q, d) = \begin{cases} OP = (p + h)d - (c + h)Q & \text{for } d \le Q \\ UP = (p - c + s)Q - sd & \text{for } d > Q \end{cases}$$

where s is the unit shortage cost. Therefore the fuzzy overstock profit and fuzzy understock profit are respectively given by

$$\widetilde{OP}(Q, \widetilde{D}) = (p + h)\widetilde{D} - (c + h)Q$$

along with the membership function (refer to Fig. 1)

$$\mu_{\widetilde{OP}}(\rho) = \begin{cases} \sup\limits_{\rho=(p+h)d-(c+h)Q} \{\mu_{\widetilde{D}}(d)\} & \text{for } d \le Q \\ 0 & \text{for } d > Q \end{cases}$$

and

$$\widetilde{UP}(Q, \widetilde{D}) = (p - c + s)Q - s\widetilde{D}$$

along with the membership function (refer to Fig. 2)

$$\mu_{\widetilde{UP}}(\rho) = \begin{cases} \sup\limits_{\rho=(p-c+s)Q-sd} \{\mu_{\widetilde{D}}(d)\} & \text{for } d \ge Q \\ 0 & \text{for } d < Q. \end{cases}$$

Obviously the resultant profit function becomes a fuzzy quantity $\tilde{P}(Q, \tilde{D})$ and without loss of generality it can be constructed as

$$\widetilde{P}(Q, \widetilde{D}) = \widetilde{OP}(Q, \widetilde{D}) \cup \widetilde{UP}(Q, \widetilde{D})$$

with the membership function

$$\mu_{\widetilde{P}}(\rho) = \max\{\mu_{\widetilde{OP}}(\rho), \mu_{\widetilde{UP}}(\rho)\}.$$

To determine the optimal order quantity one can maximize $\tilde{P}(Q, \tilde{D})$ using ranking fuzzy numbers. The expected optimal order quantity (Q^*) can be determined by optimizing the expected value of the resultant profit function [5,6] using the interval-valued expected mean of a fuzzy number or for a better optimization using the possibilistic mean value of a fuzzy number.

3. Modelling with reordering strategy under fuzzy demand

In this section, we consider a single-period products inventory model with reordering strategy considering fuzzy demand. Suppose the shopkeeper has the opportunity to reorder during the middle of the season. Assuming that there is no option for substitution between the products we propose the reordering strategy for an individual item and develop the model for profit maximization. We divide the finite planning horizon into two slots and the customer demand in each slot is considered as a fuzzy number. First, the DM is likely to determine the individual order quantity required for both the slots by optimizing the profit function incurred in each slot. We give the decision flow of the strategy as follows:

Step 1. Choose the expert's choice/guess for the first-slot demand and the second-slot demand as $\tilde{D}_1$ and $\tilde{D}_2$, respectively.

Step 2. Decision about the initial order quantity (Q_1^*) to be purchased at the beginning of the season for slot 1 (given in Section 3.1).

Step 3. Decision about the second order quantity (Q_2^*) for slot 2 without using the leftover items from slot 1 (given in Section 3.2).

Step 4. Let Q_1^+ be the number of leftover items after the end of slot 1; then Q_1^+ is defined as $Q_1^+ = \max\{0,\ Q_1^* - d_1\}$, where d_1 is the actual observation of demand during slot 1. Therefore the optimal order quantity for the second slot to be purchased at the beginning of this slot is defined as follows:

$$Q_2^{\text{opt}} = \begin{cases} 0 & \text{for } Q_1^+ \geq Q_2^* \\ Q_2^* - Q_1^+ & \text{for } Q_1^+ < Q_2^* \\ Q_2^* & \text{for } Q_1^+ = 0. \end{cases} \tag{1}$$

Since, in most cases the DM needs to know the likely Q_i-values $(\forall i = 1, 2)$, before the beginning of the season, the expected order quantity to be purchased during the whole season is Q_1^* plus Q_2^{opt} (given in Section 3.3). As an initial analysis of this fuzzy model we consider the replenishment as instantaneous along with a negligible lead-time. Now let us derive the resultant profit to be predicted for the whole season. If RP is the resultant profit function combining both the slots, for different situations as defined in Eq. (1) different RP values can be determined as follows:

Situation 1. When the leftover item from slot 1 is more than the required order quantity of slot 2, obviously there is no need to purchase items for slot 2. Hence, the resultant profit is given by

$$\text{RP} = \begin{cases} (p+\tau)d_1 - (c+\tau)Q_1^* + (p+h)d_2 - hQ_1^+ & \text{for } d_2 \leq Q_1^+ \\ (p+\tau)d_1 - (c+\tau)Q_1^* + (p+s_2)Q_1^+ - s_2 d_2 & \text{for } d_2 > Q_1^+. \end{cases}$$

Situation 2. When the leftover items from slot 1 is less than the required order quantity of slot 2 then the optimal order quantity to be purchased for slot 2 is $Q_2^{\text{opt}} = Q_2^* - Q_1^+$ and hence the resultant profit is given by

$$\text{RP} = \begin{cases} (p-c+\tau)d_1 - \tau Q_1^* + (p+h)d_2 - (c+h)Q_2^* & \text{for } d_2 \leq Q_2^* \\ (p-c+\tau)d_1 - \tau Q_1^* + (p-c+s_2)Q_2^* - s_2 d_2 & \text{for } d_2 > Q_2^*. \end{cases}$$

Situation 3. When $Q_1^+ = 0$, i.e., if there is any shortage during slot 1, then $Q_2^{\text{opt}} = Q_2^*$;

$$\text{RP} = \begin{cases} (p-c+s_1)Q_1^* - s_1 d_1 + (p+h)d_2 - (c+h)Q_2^* & \text{for } d_2 \leq Q_2^* \\ (p-c+s_1)Q_1^* - s_1 d_1 + (p-c+s_2)Q_2^* - s_2 d_2 & \text{for } d_2 > Q_2^*. \end{cases}$$

Combining above three individual resultant profits we can find the expected resultant profit. Since customer demand during each slot is prescribed imprecisely, total resultant profit becomes a fuzzy quantity, $\widetilde{\mathrm{RP}}$ (say). The detailed derivation of the expected resultant profit is in Section 3.4.

Below, in the following subsections, we first exploit the profit functions for the two slots individually, corresponding to the fuzzy demands $\tilde{D}_1$ and $\tilde{D}_2$, respectively, and then we employ the possibilistic mean value method to rank fuzzy numbers such that the optimal order quantity can be obtained.

3.1. Calculation of optimal order quantity (Q_1^*) for slot 1

Let us formulate a dummy profit function for this slot to determine the optimal order quantity Q_1^* to be purchased at the beginning of the season. When the actual observation of demand during slot 1 is d_1, for a specified order quantity Q_1 the profit incurred during this session can be described as

$$P_1(Q_1, d_1) = \begin{cases} OP_1 = (p - c + \tau)d_1 - \tau Q_1 & \text{for } d_1 \le Q_1 \\ UP_1 = (p - c + s_1)Q_1 - s_1 d_1 & \text{for } d_1 > Q_1. \end{cases}$$

We may note here that this profit function is a particular type that is applicable to situations where the DM is concerned about the profit earned in this session only. Further, the profit function may differ as per the DM's choice as may be seen in the choice for the determination of Q_2^* in Section 3.2.

The incorporation of τ in the profit function above is to take care of any disappointment charge or any possible loss of interest for extra investment per unit and also the credibility preference to the overstock profit.

Since the demand is fuzzy, the profit function associated with each order quantity Q_1 is also fuzzy. In other words, the imprecise demand $\tilde{D}_1$ in slot 1 causes either a fuzzy overstock profit ($\widetilde{OP}_1$) or a fuzzy understock profit ($\widetilde{UP}_1$) and hence the resultant profit function also becomes a fuzzy quantity $\tilde{P}_1(Q_1, \tilde{D}_1)$ and is given by

$$\widetilde{P}_1(Q_1, \widetilde{D}_1) = \widetilde{OP}_1(Q_1, \widetilde{D}_1) \cup \widetilde{UP}_1(Q_1, \widetilde{D}_1).$$

Obviously, the membership function of $\tilde{P}_1(Q_1, \tilde{D}_1)$ is of the same type, as pointed out in Section 2.3. Since $\tilde{P}_1(Q_1, \tilde{D}_1)$ is a fuzzy quantity, it cannot be directly maximized. Thus we use the possibilistic mean value of a fuzzy number to rank fuzzy numbers. Using α-level sets and the possibilistic mean value method, we find out the required Q_1^* by maximizing the mean or expected value of this fuzzy profit function. For this we need to know the α-cut of the fuzzy profit function $\tilde{P}_1(Q_1, \tilde{D}_1)$ for $\forall \alpha \in [0, 1]$.

If $P_1(Q_1, \alpha)$ is the α-level set of $\tilde{P}_1(Q_1, \tilde{D}_1)$, then it can be derived as follows:

$$\begin{aligned} P_1(Q_1, \alpha) &= OP_1(Q_1, \alpha) \cup UP_1(Q_1, \alpha) \text{ [19]} \\ &= [P_{1,L}(Q_1, \alpha), P_{1,R}(Q_1, \alpha)] \\ &= [\min\{OP_{1,L}(Q_1, \alpha), UP_{1,L}(Q_1, \alpha)\}, \max\{OP_{1,R}(Q_1, \alpha), UP_{1,R}(Q_1, \alpha)\}]. \end{aligned}$$

The possibilistic mean value of fuzzy profit function $\tilde{P}_1(Q_1, \tilde{D}_1)$ is given by

$$\overline{M}(\tilde{P}_1) = \int_0^1 \alpha(P_{1,L}(\alpha) + P_{1,R}(\alpha))\mathrm{d}\alpha.$$

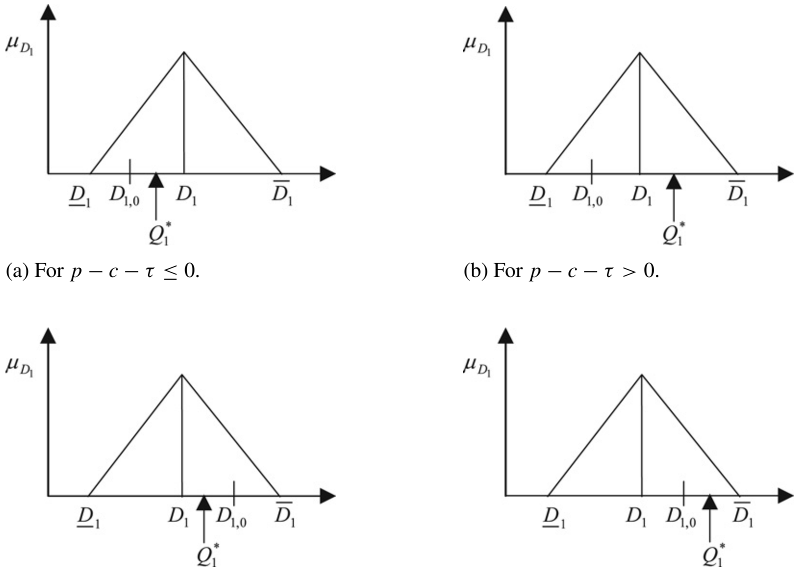
Let l_1 $(= D_1 - \underline{D}_1)$ and r_1 $(= \overline{D}_1 - D_1)$ be the left and right spreads of fuzzy demand $\tilde{D}_1$, respectively. Now to find the minimum of $OP_{1,L}$ and $UP_{1,L}$, let us define $D_{1,0} = \{(p - c + \tau)\underline{D}_1 + s_1\overline{D}_1\}/(p - c + s_1 + \tau)$ at $\alpha = 0$ so that $OP_{1,L}(Q_1, 0) \ge UP_{1,L}(Q_1, 0) \Leftrightarrow Q_1 \le D_{1,0}$.

The solution procedure for determining Q_1^* is detailed in Appendix A. We present a summary of the results in the following two cases according to the position of Q_1 in $[\underline{D}_1, \overline{D}_1]$.

Case 1. For $\underline{D}_1 \le Q_1 \le D_1$, i.e., Q_1 lies in the range between $\underline{D}_1$ and D_1.

(a) When $s_1 r_1 - l_1(p - c + \tau) > 0$ then $D_{1,0} > D_1$ and Q_1^* does not lie between $\underline{D}_1$ and D_1.
(b) When $s_1 r_1 - l_1(p - c + \tau) \le 0$ then $D_{1,0} \le D_1$ and $Q_1^* \in [D_{1,0}, D_1]$, satisfying

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv (p - c + s_1) - \frac{1}{2}[(p - c + s_1 + \tau)\alpha_1^2 + s_1\{L(Q_1^*)\}^2] = 0 \tag{2}$$



(a) For $p - c - \tau \le 0$.

(b) For $p - c - \tau > 0$.

(c) For $p - c - \tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] \le 0$.

(d) For $p - c - \tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] > 0$.

Fig. 3. Different positions of Q_1^* in $[\underline{D}_1, \overline{D}_1]$.

subject to the condition $p - c - \tau \le 0$ (α_1 is determined from $OP_{1,L}(Q_1, \alpha) = UP_{1,L}(Q_1, \alpha)$). Fig. 3 shows the possible positions of Q_1^* in $[\underline{D}_1, \overline{D}_1]$.

Case 2. For $D_1 \le Q_1 \le \overline{D}_1$, i.e., Q_1 lies in the range between D_1 and $\overline{D}_1$.

(c) When $s_1 r_1 - l_1(p - c + \tau) > 0$ then $D_{1,0} > D_1$.

If $Q_1^* \in [D_1, D_{1,0}]$ then the optimal Q_1^* is given by

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv -\tau + \frac{1}{2}[(p - c + s_1 + \tau)\alpha_1^2 + (p - c + \tau)\{R(Q_1^*)\}^2] = 0 \tag{3}$$

subject to the condition $p - c - \tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] \le 0$. Otherwise $Q_1^* \in [D_{1,0}, \overline{D}_1]$, satisfying

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv -\tau + \frac{1}{2}(p - c + \tau)\{R(Q_1^*)\}^2 = 0. \tag{4}$$

(d) When $s_1 r_1 - l_1(p - c + \tau) \le 0$ then $D_{1,0} \le D_1$ and $Q_1^* \in [D_1, \overline{D}_1]$, satisfying Eq. (4) subject to $p - c - \tau > 0$.

3.2. Calculation of optimal order quantity (Q_2^*) for slot 2

To find the optimal order quantity Q_2^* individually, we need to assume slot 2 as an independent stand-alone period, i.e., no leftover items are supplied from slot 1. If d_2 is the actual demand during slot 2 then for a given order quantity Q_2, the profit function can be constructed as

$$P_2(Q_2, d_2) = \begin{cases} OP_2 = (p + h)d_2 - (c + h)Q_2 & \text{for } d_2 \le Q_2 \\ UP_2 = (p - c + s_2)Q_2 - s_2 d_2 & \text{for } d_2 > Q_2. \end{cases}$$

Defining $\tilde{P}_2(Q_2, \tilde{D}_2)$ as the fuzzy profit function associated with fuzzy demand $\tilde{D}_2$ in slot 2, if $P_2(Q_2, \alpha)$ is the α-level set of $\tilde{P}_2(Q_2, \tilde{D}_2)$, we get the subsequent results concerning the optimal order quantity Q_2^* and $\overline{M}(\tilde{P}_2)$, the possibilistic mean value of $\tilde{P}_2(Q_2, \tilde{D}_2)$. The following notation and results come from the understanding of the derivation of Q_1^* as deduced in the previous Section 3.1.

(a) When $s_2r_2 - l_2(p+h) \leq 0$ or equivalently $D_{2,0} = \{(p+h)\underline{D}_2 + s_2\overline{D}_2\}/(p+s_2+h) \leq D_2$ then $Q_2^* \in [D_{2,0}, D_2]$ and the possibilistic mean value $\overline{M}(\tilde{P}_2)$ is given by

$$\begin{aligned}\overline{M}(\tilde{P}_2) &= (p-c+s_2)Q_2 - \frac{1}{2}[(p+s_2+h)\alpha_2^2 + s_2\{L(Q_2)\}^2]Q_2 \\ &\quad + (p+h)\int_0^{\alpha_2} \alpha D_{2,L}(\alpha)\mathrm{d}\alpha - s_2\int_{\alpha_2}^1 \alpha D_{2,R}(\alpha)\mathrm{d}\alpha - s_2\int_{L(Q_2)}^1 \alpha D_{2,L}(\alpha)\mathrm{d}\alpha \qquad (5)\end{aligned}$$

where $\alpha_2 = [(p+s_2+h)Q_2 - \{(p+h)\underline{D}_2 + s_2\overline{D}_2\}]/\{(p+h)l_2 - s_2r_2\} < L(Q_2)$.

Therefore the optimal Q_2^* is derived using the equation

$$\frac{\partial\overline{M}}{\partial Q_2}(Q_2^*) \equiv (p-c+s_2) - \frac{1}{2}[(p+s_2+h)\alpha_2^2 + s_2\{L(Q_2^*)\}^2] = 0 \qquad (6)$$

subject to the condition $p - 2c - h \leq 0$. Again if $p - 2c - h > 0$ then the optimal Q_2^* lies between D_2 and $\overline{D}_2$. In this case, $\overline{M}(\tilde{P}_2)$ is given by

$$\overline{M}(\tilde{P}_2) = -(c+h)Q_2 + \frac{1}{2}[(p+h)\{R(Q_2)\}^2]Q_2 + (p+h)\left[\int_0^1 \alpha D_{2,L}(\alpha)\mathrm{d}\alpha + \int_{R(Q_2)}^1 \alpha D_{2,R}(\alpha)\mathrm{d}\alpha\right] \qquad (7)$$

along with

$$\frac{\partial\overline{M}}{\partial Q_2}(Q_2^*) \equiv -(c+h) + \frac{1}{2}[(p+h)\{R(Q_2^*)\}^2] = 0. \qquad (8)$$

(b) When $s_2r_2 - l_2(p+h) > 0$ or equivalently $D_{2,0} > D_2$ then $Q_2^* \in [D_2, D_{2,0}]$.

In this case, $\overline{M}(\tilde{P}_2)$ is given by

$$\begin{aligned}\overline{M}(\tilde{P}_2) &= -(c+h)Q_2 + \frac{1}{2}[(p+s_2+h)\alpha_2^2 + (p+h)\{R(Q_2)\}^2]Q_2 \\ &\quad - s_2\int_0^{\alpha_2} \alpha D_{2,R}(\alpha)\mathrm{d}\alpha + (p+h)\left[\int_{\alpha_2}^1 \alpha D_{2,L}(\alpha)\mathrm{d}\alpha + \int_{R(Q_2)}^1 \alpha D_{2,R}(\alpha)\mathrm{d}\alpha\right] \qquad (9)\end{aligned}$$

along with

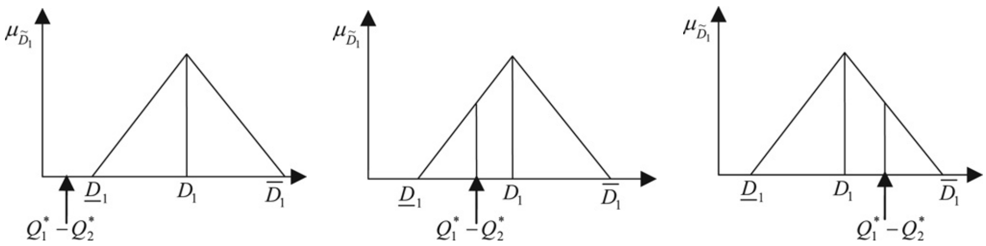
$$\frac{\partial\overline{M}}{\partial Q_2}(Q_2^*) \equiv -(c+h) + \frac{1}{2}[(p+s_2+h)\alpha_2^2 + (p+h)\{R(Q_2^*)\}^2] = 0 \qquad (10)$$

subject to the condition $-(c+h) + (p+h)\{(\overline{D}_2 - D_{2,0})/(\overline{D}_2 - D_2)\}^2 \leq 0$. Otherwise, Q_2^* lies between $D_{2,0}$ and $\overline{D}_2$ where $\overline{M}(\tilde{P}_2)$ and Q_2^* can be computed as defined in Eqs. (7) and (8), respectively.

3.3. Optimal expected total order quantity

To predict the total expected order quantity for the whole season we first derive the expected order quantity at the beginning of the second slot. Note that Q_2^{opt} is determined not at the start of slot 1, but at the end of slot 1 after the leftover value Q_1^+ is observed. However, $E(Q_2^{\text{opt}})$ can be computed to calculate the total expected order quantity for the whole season/period. Clearly, the expected optimal order quantity during the second slot depends upon the leftover items that arise after the end of the first slot. The optimal policy defined in Eq. (1) thus can be written as

$$Q_2^{\text{opt}} = \begin{cases} 0 & \text{for } \underline{D}_1 \leq d_1 \leq Q_1^* - Q_2^* \\ Q_2^* - Q_1^+ & \text{for } \max\{\underline{D}_1, Q_1^* - Q_2^*\} < d_1 < Q_1^* \\ Q_2^* & \text{for } Q_1^* \leq d_1 \leq \overline{D}_1. \end{cases} \qquad (11)$$



(a) When $Q_1^* - Q_2^* < \underline{D}_1$. (b) When $\underline{D}_1 \le Q_1^* - Q_2^* \le D_1$. (c) When $Q_1^* - Q_2^* > D_1$.

Fig. 4. Different positions of $Q_1^* - Q_2^*$.

Since the demand during slot 1 is linguistic in nature, for each order quantity Q_1, Q_1^+ is also fuzzy and consequently the decision variable during the second slot becomes a fuzzy quantity $\tilde{Q}_2^{\text{opt}}$. Let $Q_2^{\text{opt}}(\alpha)$ be its α-level set; then the expected optimal order quantity to be purchased actually during the mid-season is given by

$$\overline{M}(\tilde{Q}_2^{\text{opt}}) = \int_0^1 \alpha(Q_{2,L}^{\text{opt}}(\alpha) + Q_{2,R}^{\text{opt}}(\alpha))\mathrm{d}\alpha \tag{12}$$

where $Q_2^{\text{opt}}(\alpha) = [Q_{2,L}^{\text{opt}}(\alpha), Q_{2,R}^{\text{opt}}(\alpha)]$ is given in the following according to the position of $Q_1^* - Q_2^*$ in $[\underline{D}_1, \overline{D}_1]$ (refer to Fig. 4).

Condition 1. When $Q_1^* - Q_2^* < \underline{D}_1$ then if Q_1^* lies in the range between $\underline{D}_1$ and D_1 we have

$$Q_2^{\text{opt}}(\alpha) = \begin{cases} [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^*] & \text{for } \alpha \le L(Q_1^*) \\ [Q_2^*, Q_2^*] & \text{for } \alpha > L(Q_1^*) \end{cases}$$

and if Q_1^* lies between D_1 and $\overline{D}_1$ then

$$Q_2^{\text{opt}}(\alpha) = \begin{cases} [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^*] & \text{for } \alpha \le R(Q_1^*) \\ [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^* - Q_1^* + D_{1,R}(\alpha)] & \text{for } \alpha > R(Q_1^*). \end{cases}$$

Condition 2. When $Q_1^* - Q_2^* \ge \underline{D}_1$ then if Q_1^* lies in the range between $\underline{D}_1$ and D_1 we have

$$Q_2^{\text{opt}}(\alpha) = \begin{cases} [0, Q_2^*] & \text{for } \alpha \le L(Q_1^* - Q_2^*) \\ [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^*] & \text{for } L(Q_1^* - Q_2^*) < \alpha \le L(Q_1^*) \\ [Q_2^*, Q_2^*] & \text{for } \alpha > L(Q_1^*). \end{cases}$$

Again if Q_1^* lies between D_1 and $\overline{D}_1$ then for $L(Q_1^* - Q_2^*) < R(Q_1^*)$ we have

$$Q_2^{\text{opt}}(\alpha) = \begin{cases} [0, Q_2^*] & \text{for } \alpha \le L(Q_1^* - Q_2^*) \\ [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^*] & \text{for } L(Q_1^* - Q_2^*) < \alpha \le R(Q_1^*) \\ [Q_2^* - Q_1^* + D_{1,L}(\alpha), Q_2^* - Q_1^* + D_{1,R}(\alpha)] & \text{for } \alpha > R(Q_1^*). \end{cases}$$

Similarly, another α-level set of $\tilde{Q}_2^{\text{opt}}$ can be derived when $L(Q_1^* - Q_2^*) \ge R(Q_1^*)$ or $Q_1^* - Q_2^* > D_1$.

Hence the total expected optimal order quantity combining slot 1 and slot 2 is determined by

$$Q_1^* + \text{Expected } \tilde{Q}_2^{\text{opt}} = Q_1^* + \overline{M}(\tilde{Q}_2^{\text{opt}}).$$

3.4. Expected resultant profit

Here we determine the expected resultant profit combining slot 1 and slot 2. It is common to model the profit functions according to the situations as defined in the optimal policy (1); the parameters d_1 and d_2 are uncertain and we will substitute fuzzy numbers $\tilde{D}_1$ and $\tilde{D}_2$ for them, respectively. Consequently, the resultant profit function becomes a fuzzy quantity $\widetilde{\text{RP}}$ (say). To find the expected value of the resultant profit $\widetilde{\text{RP}}$, we first need to find the logical union of the individual resultant profit functions.

Mathematically, it is determined by

$$\widetilde{\mathrm{RP}} = \widetilde{\mathrm{RP}}_{\text{Situation 1}} \cup \widetilde{\mathrm{RP}}_{\text{Situation 2}} \cup \widetilde{\mathrm{RP}}_{\text{Situation 3}}.$$

It is hard to find the exact shape of the membership function of $\widetilde{\mathrm{RP}}$. In doing so, for a given fuzzy demand $\tilde{D}_2$ during slot 2 first find the expected profit $\overline{M}(\tilde{P}_2)$ and then find several α-level sets of $\widetilde{\mathrm{RP}}$ using the α-cut of $\tilde{D}_1$ such that the expected resultant profit can be found. Thus if $\mathrm{RP}(\alpha) = [\mathrm{RP}_L(\alpha), \mathrm{RP}_R(\alpha)]$ is the α-cut of this fuzzy resultant profit function, then its possibilistic mean value is given by

$$\overline{M}(\widetilde{\mathrm{RP}}) = \int_0^1 \alpha(\mathrm{RP}_L(\alpha) + \mathrm{RP}_R(\alpha))\mathrm{d}\alpha.$$

The following α-cuts will clarify how to determine the expected resultant profit combining the three situations.
Condition 1. When $Q_1^* - Q_2^* < \underline{D}_1$ then *Situation* 1 will never happen and $\mathrm{RP}(\alpha)$ can be computed as follows:

For Q_1^* lying in the range between $\underline{D}_1$ and D_1 we have

$$\mathrm{RP}_L(\alpha) = \begin{cases} \min\{(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1^* + \overline{M}(\tilde{P}_2), (p-c+s_1)Q_1^* - s_1 Q_1^* + \overline{M}(\tilde{P}_2)\} & \text{for } \alpha \le L(Q_1^*) \\ (p-c+s_1)Q_1^* - s_1 D_{1,R}(\alpha) + \overline{M}(\tilde{P}_2) & \text{for } \alpha > L(Q_1^*) \end{cases}$$

$$\mathrm{RP}_R(\alpha) = \begin{cases} \max\{(p-c+\tau)Q_1^* - \tau Q_1^* + \overline{M}(\tilde{P}_2), (p-c+s_1)Q_1^* - s_1 D_{1,L}(\alpha) + \overline{M}(\tilde{P}_2)\} & \text{for } \alpha \le L(Q_1^*) \\ (p-c+s_1)Q_1^* - s_1 D_{1,L}(\alpha) + \overline{M}(\tilde{P}_2) & \text{for } \alpha > L(Q_1^*) \end{cases}$$

and if Q_1^* lies between D_1 and $\overline{D}_1$ then

$$\mathrm{RP}_L(\alpha) = \begin{cases} \min\{(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1^* + \overline{M}(\tilde{P}_2), (p-c+s_1)Q_1^* - s_1 D_{1,R}(\alpha) + \overline{M}(\tilde{P}_2)\} & \text{for } \alpha \le R(Q_1^*) \\ (p-c+\tau)D_{1,L}(\alpha) - \tau Q_1^* + \overline{M}(\tilde{P}_2) & \text{for } \alpha > R(Q_1^*) \end{cases}$$

$$\mathrm{RP}_R(\alpha) = \begin{cases} \max\{(p-c+\tau)Q_1^* - \tau Q_1^* + \overline{M}(\tilde{P}_2), (p-c+s_1)Q_1^* - s_1 Q_1^* + \overline{M}(\tilde{P}_2)\} & \text{for } \alpha \le R(Q_1^*) \\ (p-c+\tau)D_{1,R}(\alpha) - \tau Q_1^* + \overline{M}(\tilde{P}_2) & \text{for } \alpha > R(Q_1^*) \end{cases}$$

where $\overline{M}(\tilde{P}_2)$ is given by either Eqs. (5) and (7) or (9) according to the position of Q_2^*.
Condition 2. When $Q_1^* - Q_2^* \ge \underline{D}_1$ then $\mathrm{RP}(\alpha)$ can be computed as follows:

For Q_1^* lying in the range between $\underline{D}_1$ and D_1 we have

$$\mathrm{RP}_L(\alpha) = \begin{cases} \min\{(p+\tau)D_{1,L}(\alpha) - (c+\tau)Q_1^* + \overline{M}(\tilde{P}_d), (p-c+\tau)(Q_1^* - Q_2^*) - \tau Q_1^* + \overline{M}(\tilde{P}_2), \\ \quad (p-c+s_1)Q_1^* - s_1 D_{1,R}(\alpha) + \overline{M}(\tilde{P}_2)\} \quad \text{for } \alpha \le L(Q_1^* - Q_2^*) \\ \min\{(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1^* + \overline{M}(\tilde{P}_2), (p-c+s_1)Q_1^* - s_1 D_{1,R}(\alpha) + \overline{M}(\tilde{P}_2)\} \\ \quad \text{for } L(Q_1^* - Q_2^*) < \alpha \le L(Q_1^*) \\ (p-c+s_1)Q_1^* - s_1 D_{1,R}(\alpha) + \overline{M}(\tilde{P}_2) \quad \text{for } \alpha > L(Q_1^*) \end{cases}$$

$$\mathrm{RP}_R(\alpha) = \begin{cases} \max\{(p+\tau)(Q_1^* - Q_2^*) - (c+\tau)Q_1^* + \overline{M}(\tilde{P}_d), (p-c)Q_1^* + \overline{M}(\tilde{P}_2)\} \quad \text{for } \alpha \le L(Q_1^* - Q_2^*) \\ (p-c)Q_1^* + \overline{M}(\tilde{P}_2) \quad \text{for } L(Q_1^* - Q_2^*) < \alpha \le L(Q_1^*) \\ (p-c+s_1)Q_1^* - s_1 D_{1,L}(\alpha) + \overline{M}(\tilde{P}_2) \quad \text{for } \alpha > L(Q_1^*) \end{cases}$$

where $\overline{M}(\tilde{P}_d)$ is computed in Appendix B.

Similarly, all possible $\mathrm{RP}(\alpha)$ can be obtained when $Q_1^* > D_1$ and using these α-cuts the expected resultant profit can be determined.

Thus we conclude that, if there is an opportunity for reordering, then the optimal order quantity and the expected profit can be found in this fuzzy environment. In the following section we give some practical numerical examples to illustrate the reordering strategy.

4. Numerical examples

Example 1. Suppose in the winter season, a spot seller stores woolen materials for the season. Total seasonal time can be divided into two slots each carries two months. The seller collects the demand information from the experts

Table 1
Results using the reordering strategy

τ	Q_1^*	Q_2^*	$\overline{M}(\tilde{P}_2)$	$\overline{M}(\tilde{Q}_2^{\text{opt}})$	Expected order quantity $Q_1^* + \overline{M}(\tilde{Q}_2^{\text{opt}})$	Total expected profit $\overline{M}(\widetilde{\text{RP}})$
1.00	3184	1955	6898.41	1750	4934	18 609.23
0.50	3264	1955	6898.41	1682	4946	18 726.97
0.00	3500	1955	6898.41	1445	4955	18 898.41
When the profit function in slot 1 is the same as that in slot 2						
1.00	2977	1955	6898.41	1883	4860	18 233.84
0.50	2979	1955	6898.41	1882	4861	18 254.01
0.00	2982	1955	6898.41	1881	4863	18 260.40

Table 2
Parametric values for different items

Items	c	p	s_1	s_2	h	Demand information		
						During slot 1	During slot 2	For whole season
1	8	12	6	4	2	(1300, 1500, 1700)	(800, 1000, 1200)	(2100, 2500, 2900)
2	9	13	6	4	2	(800, 1000, 1200)	(500, 700, 900)	(1300, 1700, 2100)
3	10	14	6	4	2	(700, 800, 900)	(400, 500, 600)	(1100, 1300, 1500)

Table 3
For the two-order case

Items	τ	Q_1^*	Q_2^*	$\overline{M}(\tilde{P}_2)$	$\overline{M}(\tilde{Q}_2^{\text{opt}})$	Expected order quantity $Q_1^* + \overline{M}(\tilde{Q}_2^{\text{opt}})$	Total expected profit $\overline{M}(\widetilde{\text{RP}})$
1	0.16	1644	982	3559.36	837	2481	9533.28
2	0.18	1141	679	2335.25	537	1678	6306.29
3	0.20	869	488	1756.19	418	1287	4940.30

in each slot rather than the whole season. But due to the lack of information or linguistic information, imprecision occurs like '*demand is about d*'. Let, $c = 8$, $p = 12$, $h = 2$, $s_1 = 6$ and $s_2 = 4$.

The customer demands in each slot are forecast as triangular fuzzy numbers $\tilde{D}_1 = (2500, 3000, 3500)$ and $\tilde{D}_2 = (1500, 2000, 2500)$, respectively. Different results for different values of τ are presented in Table 1.

It is observed from Table 1 that whenever the profit function constructed for determining Q_1^* in Section 3.1 is of type Section 3.2 (subsection), the expected value of the total resultant profit decreases. On the other hand, the DM earns more profit if he/she decreases the value of τ.

Example 2. To compare the results of proposed reordering strategy with those obtained from the single-order case, let us consider the inventory system in multi-item environments. Suppose the shopkeeper stores three items under his/her management. Different input values for different items are given in Table 2. The disappointment charge τ for items unsold during slot 1 is considered as 2% of unit purchase cost. In Table 5, results obtained from the reordering case (Table 3) and single-ordering case (Table 4) are compared.

Therefore, the overall extra expected profit due to the second-order opportunity is $\overline{M}(\widetilde{\text{RP}}) = 1078.30$. The above examples show that when the DM decreases τ (the unsatisfied charge associated with unit unsold item after slot1), profit increases. In summary, if there is an opportunity for reorder then the management is much more profitable with this reordering strategy compared to the simple single-order case.

5. Conclusions

This paper deals with the maximizing the expected profit of a retailer facing fuzzy demand under a single-period framework, where the retailer has an opportunity to reorder once during the period. Though the reordering strategy

Table 4
For the one-order case

Items	Shortage cost s	Order quantity Q^*	Total resultant profit $\overline{M}(\tilde{P})$
1	4	2464	9118.73
	6	2477	9100.29
2	4	1658	5870.50
	6	1672	5846.86
3	4	1276	4712.34
	6	1284	4697.76

Table 5
Comparison between the one-order case (when $s = 4$) and the two-order case

Items	One-order case		Two-order case				Initial reduction		Profit increases
	Q^*	$\overline{M}(\tilde{P})$	τ	Q_1^*	$Q_1^* + \overline{M}(\tilde{Q}_2^{\text{opt}})$	$\overline{M}(\widetilde{\text{RP}})$	Order quantity	Investment	
1	2464	9118.73	0.16	1644	2481	9533.28	820	6560.00	414.55
2	1658	5870.50	0.18	1141	1678	6306.29	517	4653.00	435.79
3	1276	4712.34	0.20	869	1287	4940.30	407	4070.00	227.96

adopted here is very complicated compared to the simple single-order case, it makes more practical sense from the managerial point of view. Several α-level sets as defined in Section 3.4 can be worked out either with any programming language or graphically. The work may be extended considering the effects of set-up cost during the second order together with delivery delay in this imprecise environment. The proposed fuzzy model of reordering strategy can also be extended further considering budget constraints [11].

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Appendix A. Details of determining Q_1^*

For determining the optimal order quantity Q_1^* as stated in Section 3.2, let us first derive the possibilistic mean value of fuzzy profit function $\tilde{P}_1(Q_1, \tilde{D}_1)$ as

$$\overline{M}(\tilde{P}_1) = \int_0^1 \alpha(P_{1,L}(\alpha) + P_{1,R}(\alpha))\mathrm{d}\alpha.$$

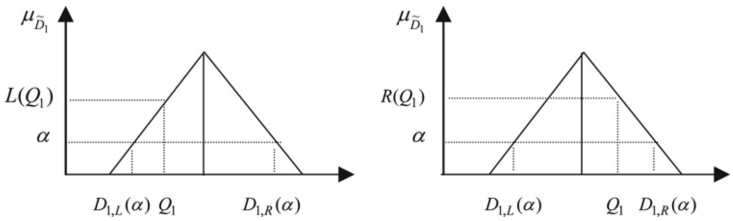
Now we optimize this $\overline{M}(\tilde{P}_1)$ with respect to Q_1 so that the optimal Q_1^* can be obtained.

Case 1. When Q_1 lies in the range between $\underline{D}_1$ and D_1 we have

$$\begin{cases} OP_1(Q_1,\alpha) = [(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, (p-c)Q_1] \\ UP_1(Q_1,\alpha) = [(p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), (p-c)Q_1] \end{cases} \quad \text{for } \alpha \le L(Q_1)$$
$$\begin{cases} OP_1(Q_1,\alpha) = \phi \\ UP_1(Q_1,\alpha) = [(p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), (p-c+s_1)Q_1 - s_1 D_{1,L}(\alpha)] \end{cases} \quad \text{for } \alpha > L(Q_1).$$

To check the minimum of $OP_{1,L}$ and $UP_{1,L}$ at $\alpha = 0$, let us take $D_{1,0} = \frac{(p-c+\tau)\underline{D}_1 + s_1\overline{D}_1}{p-c+\tau+s_1} (= D_1 + \frac{s_1r_1 - l_1(p-c+\tau)}{p-c+\tau+s_1})$ so that $D_{1,0} >$ or $\le D_1$ according as $s_1r_1 - l_1(p-c+\tau) > 0$ or ≤ 0 (see [5]). Now it can be easily verified that $Q_1 <$ or $\ge D_{1,0} \Leftrightarrow OP_{1,L}(Q_1, 0) >$ or $\le UP_{1,L}(Q_1, 0)$ (refer to Fig. 5).

Case 1a. If $s_1r_1 - l_1(p-c+\tau) > 0$ then $D_{1,0} > D_1$.



(a) When $\underline{D}_1 \le Q_1 \le D_1$. (b) When $D_1 \le Q_1 \le \overline{D}_1$.

Fig. 5. Alpha cut of $\tilde{D}_1$.

Therefore, $P_{1,L}(Q_1,\alpha) = (p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), 0 \le \alpha \le 1$;

$$P_{1,R}(Q_1,\alpha) = \begin{cases} (p-c)Q_1, & 0 \le \alpha \le L(Q_1) \\ (p-c+s_1)Q_1 - s_1 D_{1,L}(\alpha), & L(Q_1) \le \alpha \le 1. \end{cases}$$

Thus, the mean profit function $\overline{M}(\tilde{P}_1)$ is computed from

$$\begin{aligned}\overline{M}(\tilde{P}_1) &= \int_0^1 \alpha\{(p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha)\}\mathrm{d}\alpha + \int_0^{L(Q_1)} \alpha(p-c)Q_1 \mathrm{d}\alpha \\ &\quad + \int_{L(Q_1)}^1 \alpha\{(p-c+s_1)Q_1 - s_1 D_{1,L}(\alpha)\}\mathrm{d}\alpha \\ &= (p-c+s_1)Q_1 - \frac{s_1}{2}Q_1\{L(Q_1)\}^2 - s_1\int_0^1 \alpha D_{1,R}(\alpha)\mathrm{d}\alpha - s_1 \int_{L(Q_1)}^1 \alpha D_{1,L}(\alpha)\mathrm{d}\alpha.\end{aligned}$$

The first and second derivatives with respect to Q_1 are

$$\frac{\partial \overline{M}}{\partial Q_1} = (p-c+s_1) - \frac{s_1}{2}\{L(Q_1)\}^2, \qquad \frac{\partial^2 \overline{M}}{\partial Q_1^2} = -s_1 L(Q_1)L'(Q_1).$$

As $L(Q_1)$ is an increasing function, $\frac{\partial^2\overline{M}}{\partial Q_1^2} < 0$. Consequently, $\overline{M}(\tilde{P}_1)$ is concave on the interval $[\underline{D}_1, D_1]$. Now, $\frac{\partial \overline{M}(\underline{D}_1)}{\partial Q_1} = (p-c+s_1) > 0$ and $\frac{\partial \overline{M}(D_1)}{\partial Q_1} = (p-c+\frac{s_1}{2}) > 0$.

In this case, the optimal Q_1^* does not exist in the interval $[\underline{D}_1, D_1]$.

Case 1b. If $s_1 r_1 - l_1(p-c+\tau) \le 0$ then $D_{1,0} \le D_1$.

Case 1b.1 When $\underline{D}_1 \le Q_1 < D_{1,0}$ then $P_{1,L}(Q_1,\alpha)$ and $P_{1,R}(Q_1,\alpha)$ are the same as defined in Case 1a. So the optimality is not attaining here again.

Case 1b.2 When $D_{1,0} \le Q_1 \le D_1$ then to find the minimum of $OP_{1,L}$ and $UP_{1,L}$,

let $\alpha_1 = \frac{(p-c+s_1+\tau)Q_1 - [s_1\overline{D}_1 + (p-c+\tau)\underline{D}_1]}{(p-c+\tau)l_1 - s_1 r_1} (\alpha_1 < L(Q_1))$ where $OP_{1,L}(Q_1,\alpha_1) = UP_{1,L}(Q_1,\alpha_1)$.

Therefore,

$$P_{1,L}(Q_1,\alpha) = \begin{cases} (p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, & 0 \le \alpha \le \alpha_1 \\ (p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), & \alpha_1 \le \alpha \le 1 \end{cases};$$

$$P_{1,R}(Q_1,\alpha) = \begin{cases} (p-c)Q_1, & 0 \le \alpha \le L(Q_1) \\ (p-c+s_1)Q_1 - s_1 D_{1,L}(\alpha), & L(Q_1) \le \alpha \le 1. \end{cases}$$

Thus the possibilistic mean is given by

$$\begin{aligned}\overline{M}(\tilde{P}_1) &= (p-c+s_1)Q_1 - \frac{1}{2}[(p-c+s_1+\tau)\alpha_1^2 + s_1\{L(Q_1)\}^2]Q_1 \\ &\quad + (p-c+s_1)\int_0^{\alpha_1} \alpha D_{1,L}(\alpha)\mathrm{d}\alpha - s_1\int_{\alpha_1}^1 \alpha D_{1,R}(\alpha)\mathrm{d}\alpha - s_1\int_{L(Q_1)}^1 \alpha D_{1,L}(\alpha)\mathrm{d}\alpha.\end{aligned}$$

Also $\frac{\partial \overline{M}}{\partial Q_1} = (p-c+s_1) - \frac{1}{2}[(p-c+s_1+\tau)\alpha_1^2 + s_1\{L(Q_1)\}^2]$ and $\frac{\partial^2\overline{M}}{\partial Q_1^2} = -[(p-c+s_1+\tau)\alpha_1\alpha_1' + s_1 L(Q_1)L'(Q_1)]$.

Again $\frac{\partial \overline{M}(D_{1,0})}{\partial Q_1} = (p-c+s_1) - \frac{s_1}{2}(\frac{D_{1,0}-\underline{D}_1}{l_1})^2 > (p-c+\frac{s_1}{2}) > 0$,

$$\frac{\partial \overline{M}(D_1)}{\partial Q_1} = \frac{1}{2}(p-c-\tau) \le 0 \quad \text{if } p-c-\tau \le 0.$$

Hence, $\frac{\partial^2 \overline{M}}{\partial Q_1^2} < 0$ and the optimal Q_1^* is given by

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv (p-c+s_1) - \frac{1}{2}[(p-c+s_1+\tau)\alpha_1^2 + s_1\{L(Q_1^*)\}^2] = 0 \tag{A.1}$$

subject to the condition $p-c-\tau \le 0$.

Case 2. When Q_1 lies in the range between D_1 and $\overline{D}_1$ we have

$$\begin{cases} OP_1(Q_1,\alpha) = [(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, (p-c)Q_1] \\ UP_1(Q_1,\alpha) = [(p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), (p-c)Q_1] \end{cases} \quad \text{for } \alpha \le R(Q_1)$$

$$\begin{cases} OP_1(Q_1,\alpha) = [(p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, (p-c+\tau)D_{1,R}(\alpha) - \tau Q_1] \\ UP_1(Q_1,\alpha) = \phi. \end{cases} \quad \text{for } \alpha > R(Q_1).$$

Case 2a. If $s_1 r_1 - l_1(p-c+\tau) > 0$ then $D_{1,0} > D_1$.
Case 2a.1 When $D_1 \le Q_1 \le D_{1,0}$ then

$$P_{1,L}(Q_1,\alpha) = \begin{cases} (p-c+s_1)Q_1 - s_1 D_{1,R}(\alpha), & 0 \le \alpha \le \alpha_1 \\ (p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, & \alpha_1 \le \alpha \le 1 \end{cases};$$

$$P_{1,R}(Q_1,\alpha) = \begin{cases} (p-c)Q_1, & 0 \le \alpha \le R(Q_1) \\ (p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, & R(Q_1) \le \alpha \le 1. \end{cases}$$

Therefore,

$$\begin{aligned} \overline{M}(\tilde{P}_1) = & -\tau Q_1 + \frac{1}{2}[(p-c+s_1+\tau)\alpha_1^2 + (p-c+\tau)\{R(Q_1)\}^2]Q_1 \\ & - s_1\int_0^{\alpha_1} \alpha D_{1,R}(\alpha)\mathrm{d}\alpha + (p-c+\tau)\left[\int_{\alpha_1}^1 \alpha D_{1,L}(\alpha)\mathrm{d}\alpha + \int_{R(Q_1)}^1 \alpha D_{1,R}(\alpha)\mathrm{d}\alpha\right]. \end{aligned}$$

It can be shown that $\frac{\partial^2 \overline{M}}{\partial Q_1^2} < 0$ and $\frac{\partial \overline{M}(D_1)}{\partial Q_1} > 0$.

Now $\frac{\partial \overline{M}(D_{1,0})}{\partial Q_1} = -\tau + \frac{1}{2}(p-c+\tau)(\frac{\overline{D}_1 - D_{1,0}}{r_1})^2 \le 0$ if $p-c-\tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] \le 0$.

Hence, the optimal Q_1^* is given by

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv -\tau + \frac{1}{2}[(p-c+s_1+\tau)\alpha_1^2 + (p-c+\tau)\{R(Q_1^*)\}^2] = 0 \tag{A.2}$$

subject to the condition $p-c-\tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] \le 0$.
Case 2a.2 When $D_{1,0} < Q_1 \le \overline{D}_1$ then

$$P_{1,L}(Q_1,\alpha) = (p-c+\tau)D_{1,L}(\alpha) - \tau Q_1, \quad 0 \le \alpha \le 1$$

$$P_{1,R}(Q_1,\alpha) = \begin{cases} (p-c)Q_1, & 0 \le \alpha \le R(Q_1) \\ (p-c+\tau)D_{1,R}(\alpha) - \tau Q_1, & R(Q_1) \le \alpha \le 1. \end{cases}$$

Therefore, $\overline{M}(\tilde{P}_1) = -\tau Q_1 + \frac{1}{2}(p-c+\tau)\{R(Q_1)\}^2 Q_1 + (p-c+\tau)[\int_0^1 \alpha D_{1,L}(\alpha)\mathrm{d}\alpha + \int_{R(Q_1)}^1 \alpha D_{1,R}(\alpha)\mathrm{d}\alpha]$.

In this case, also $\frac{\partial^2 \overline{M}}{\partial Q_1^2} < 0$ and the optimal Q_1^* is given by

$$\frac{\partial \overline{M}}{\partial Q_1}(Q_1^*) \equiv -\tau + \frac{1}{2}(p-c+\tau)\{R(Q_1^*)\}^2 = 0 \tag{A.3}$$

subject to the condition $p - c - \tau[2(\frac{\overline{D}_1 - D_1}{\overline{D}_1 - D_{1,0}})^2 - 1] > 0$.

Case 2b. If $s_1 r_1 - l_1(p - c + \tau) \le 0$ then $D_{1,0} \le D_1$.

Here, $P_{1,L}(Q_1, \alpha)$ and $P_{1,R}(Q_1, \alpha)$ are the same as those defined in Case 2a.2.

Again $\frac{\partial \overline{M}(D_1)}{\partial Q_1} = \frac{p-c-\tau}{2} > 0$ if $p - c - \tau > 0$ and $\frac{\partial \overline{M}(\overline{D}_1)}{\partial Q_1} = -\tau < 0$.

So the optimal Q_1^* is given by Eq. (A.3) if $p - c - \tau > 0$.

Appendix B. Possibilistic mean value of $\widetilde{P_d}(G(\widetilde{Q_1^+}))$

Earlier we mentioned that d_1 is uncertain. Corresponding to the fuzzy demand $\tilde{D}_1$ instead of d_1, Q_1^+ becomes a fuzzy quantity. When $Q_1^* - Q_2^* > \underline{D}_1$, to calculate the resultant profit as defined in Section 3.4 we use the idea of a graded mean integration representation of a fuzzy number [18] to find the graded mean value of $\tilde{Q}_1^+ = \max\{0, Q_1^* - \tilde{D}_1\}$ such that the comparison $d_2 <$ or $> Q_1^+$ can be made and as a result the expected resultant profit can be found.

Thus, to avoid complexity we transform the relation $d_2 <$ or $> Q_1^+$ into $d_2 <$ or $> G(\tilde{Q}_1^+)$ where $G(\tilde{Q}_1^+)$ is given as follows:

If $Q_1^* - Q_2^* \le D_1$ then

$$Q_1^+(\alpha) = [Q_2^*, Q_1^* - D_{1,L}(\alpha)] \quad \text{for } \alpha \le L(Q_1^* - Q_2^*).$$

Thus, $G(\tilde{Q}_1^+) = \int_0^{L(Q_1^*-Q_2^*)} \alpha \left(\frac{Q_2^* + Q_1^* - D_{1,L}(\alpha)}{2}\right) \mathrm{d}\alpha \Big/ \int_0^{L(Q_1^*-Q_2^*)} \alpha \mathrm{d}\alpha$.

Again, if $Q_1^* - Q_2^* > D_1$ then

$$Q_1^+(\alpha) = \begin{cases} [Q_2^*, Q_1^* - D_{1,L}(\alpha)] & \text{for } \alpha \le R(Q_1^* - Q_2^*) \\ [Q_1^* - D_{1,R}(\alpha), Q_1^* - D_{1,L}(\alpha)] & \text{for } \alpha > R(Q_1^* - Q_2^*). \end{cases}$$

Thus,

$$G(\tilde{Q}_1^+) = \left\{ \int_0^{R(Q_1^*-Q_2^*)} \alpha \left(\frac{Q_2^* + Q_1^* - D_{1,L}(\alpha)}{2} \right) \mathrm{d}\alpha \right. \\ \left. + \int_{R(Q_1^*-Q_2^*)}^1 \alpha \left(\frac{Q_1^* - D_{1,R}(\alpha) + Q_1^* - D_{1,L}(\alpha)}{2} \right) \mathrm{d}\alpha \right\} \Big/ \int_0^1 \alpha \mathrm{d}\alpha.$$

Let us define

$$P_d(G(\tilde{Q}_1^+), d_2) = \begin{cases} (p+h)d_2 - hG(\tilde{Q}_1^+) & \text{for } d_2 \le G(\tilde{Q}_1^+) \\ (p+s_2)G(\tilde{Q}_1^+) - s_2 d_2 & \text{for } d_2 > G(\tilde{Q}_1^+). \end{cases}$$

So far we have assumed that d_2 is the actual representative of fuzzy demand $\tilde{D}_2$; consequently, the above dummy profit function $P_d(G(\tilde{Q}_1^+), d_2)$ becomes a fuzzy quantity $\tilde{P}_d(G(\tilde{Q}_1^+))$ (say). Thus, to find out the possibilistic mean value of $\tilde{P}_d$, let us first check the position of $G(\tilde{Q}_1^+)$ in $[\underline{D}_2, \overline{D}_2]$. If $G(\tilde{Q}_1^+) \in [\underline{D}_2, D_2]$ then $\overline{M}(\tilde{P}_d)$ can be determined from Eq. (5), replacing $c = 0$ and $Q_2^* = G(\tilde{Q}_1^+)$. In a similar way, if $G(\tilde{Q}_1^+) \in [D_2, \overline{D}_2]$ then $\overline{M}(\tilde{P}_d)$ will be found from either Eq. (7) or (9).

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